# Model Selection for Sequential Inference and Decision-making

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ETH zürich | Learning & Adaptive Systems

# Sequential Decision-Making & Bandits: Problem

At every step $t$

Choose actions $\boldsymbol{x}_t$

unknown reward

$$y_t = r(\boldsymbol{x}_t) + \varepsilon_t$$

obsv.noise

Receive feedback $y_t$

Repeat

image source: flaticon

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Motivation: maximize r using the fewest queries

image source: flaticon

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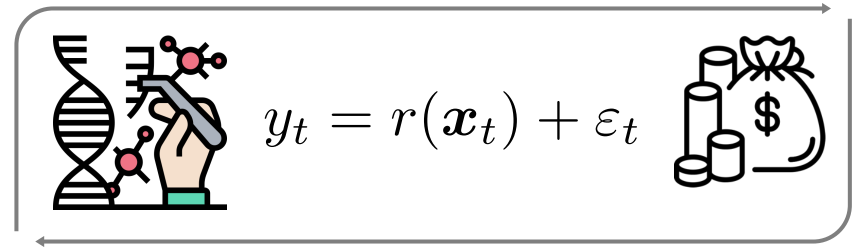


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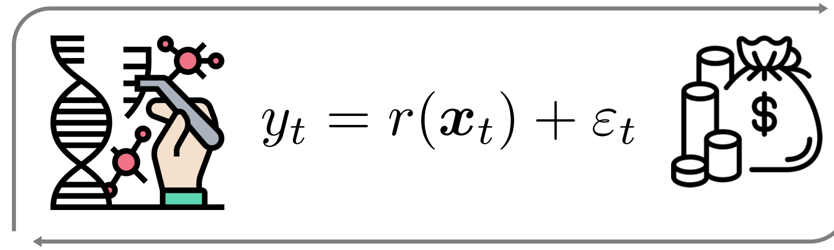


To take actions at every step:

- Estimate the reward function

based on:

Statistical model for the reward e.g. r is a linear function

history $H_{t-1} = \{(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_{t-1}, y_{t-1})\}$

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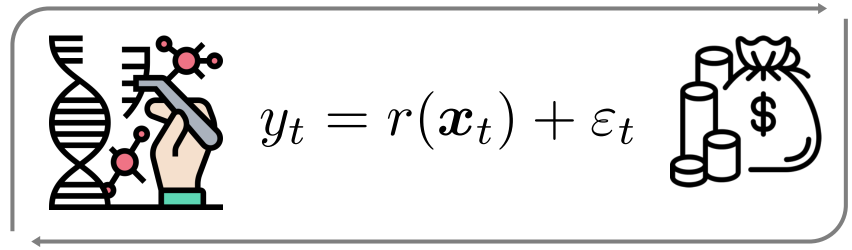


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Heavily rely on the choice of model → Model selection is key!

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Open problem: when is (efficient) online model selection possible?

[Agarwal et al. 2017]

image source: flaticon

## Online Model Selection problem

Find $j^\star$ while maximizing for the unknown $r$

$$R(T) = \sum_{t=1}^{T} r(\boldsymbol{x}^\star) - r(\boldsymbol{x}_t)$$

– Sublinear in $T$

– $\log M$

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## Online Model Selection problem

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**Why do we need to select?**
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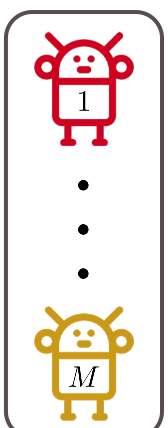


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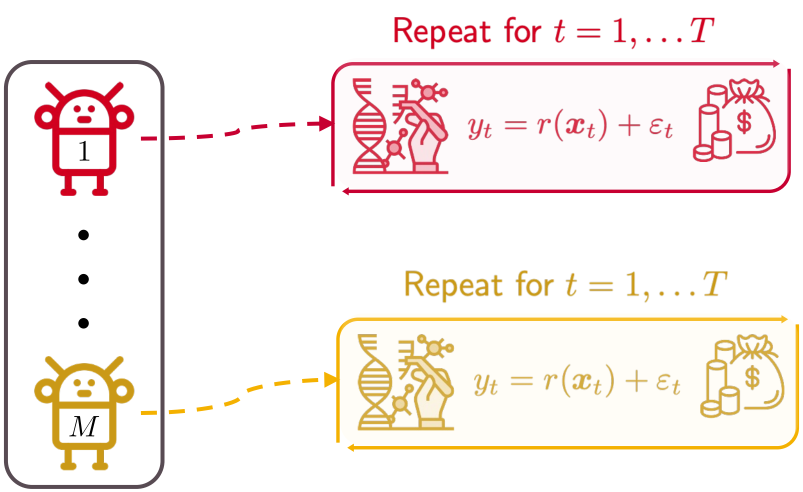


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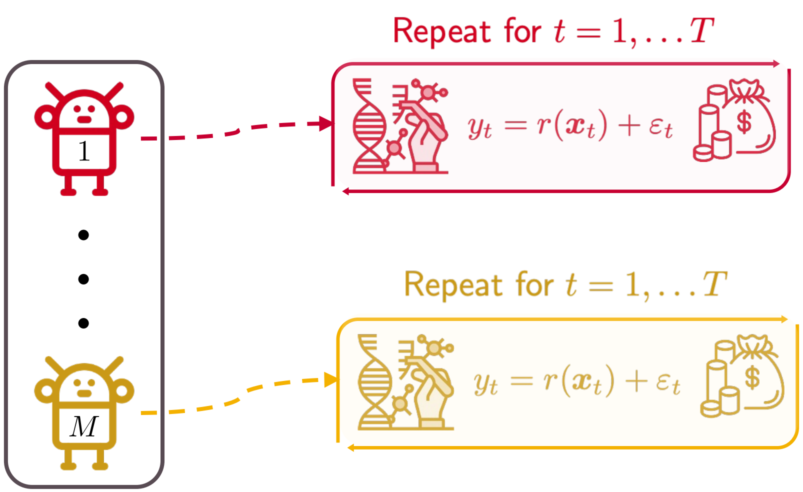


Statistically expensive
$\leftarrow\rightarrow$ High regret

$$\mathrm{poly}(M)$$

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Model Class $\{\boldsymbol{\phi}_j : \mathbb{R}^{d_0} \rightarrow \mathbb{R}^d,\ j = 1, \dots, M\}$ $\quad M \gg T$

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Is not any-time: only works if horizon $T$ is known in advance (doubling trick aside)

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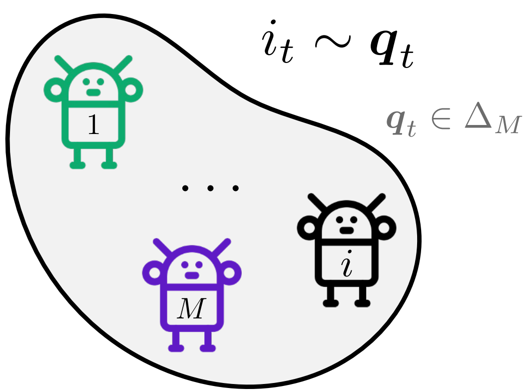
Instatiate M "agents"

- Agent j only uses $\phi_j$ to model the reward
- Has action selection strategy $p_{t,j} \in \mathcal{M}(\mathcal{X})$ which is updated at every step e.g. UCB [for those who know]

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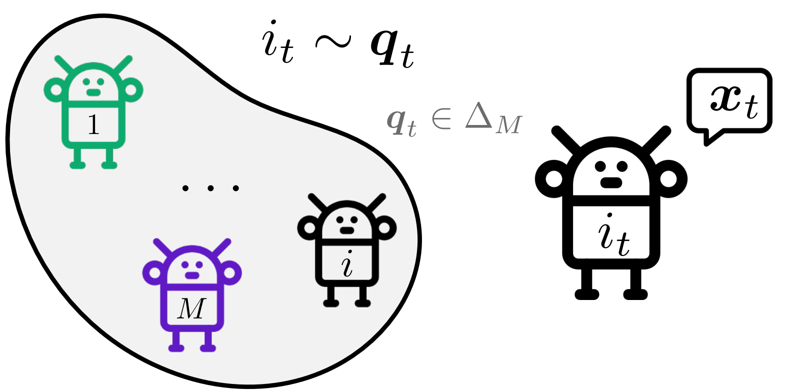


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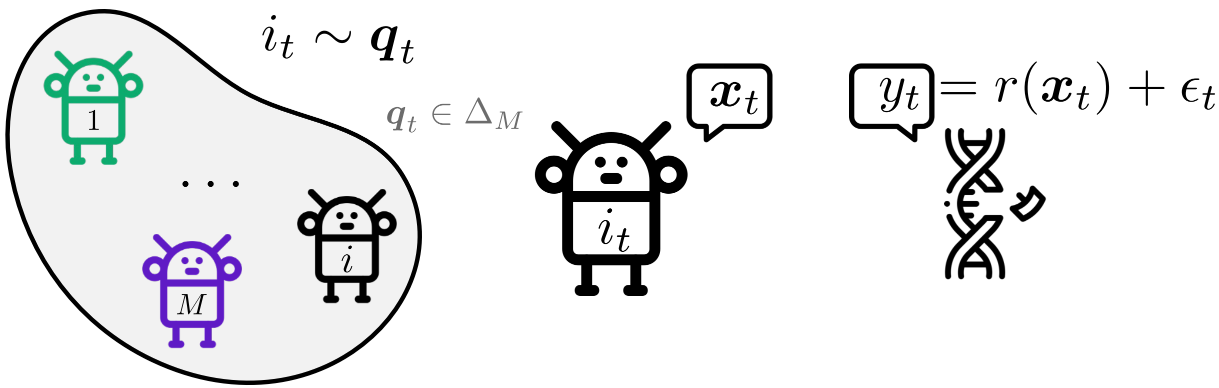


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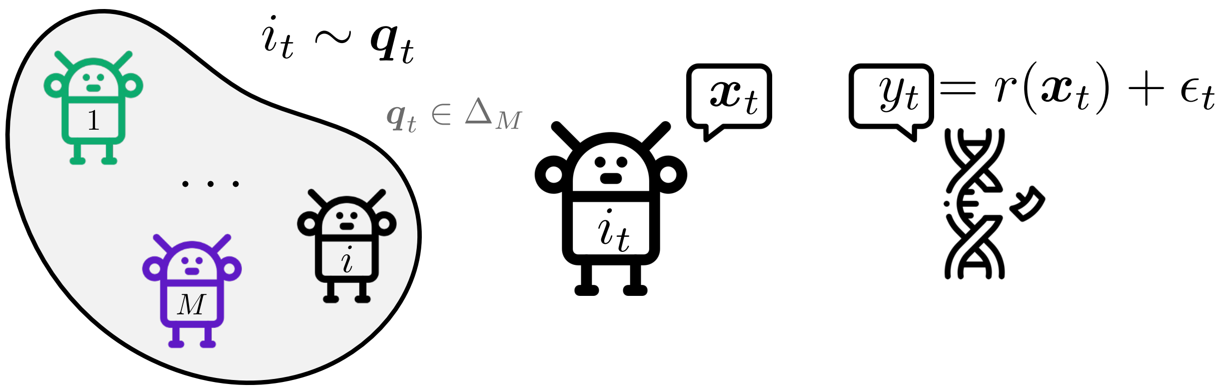


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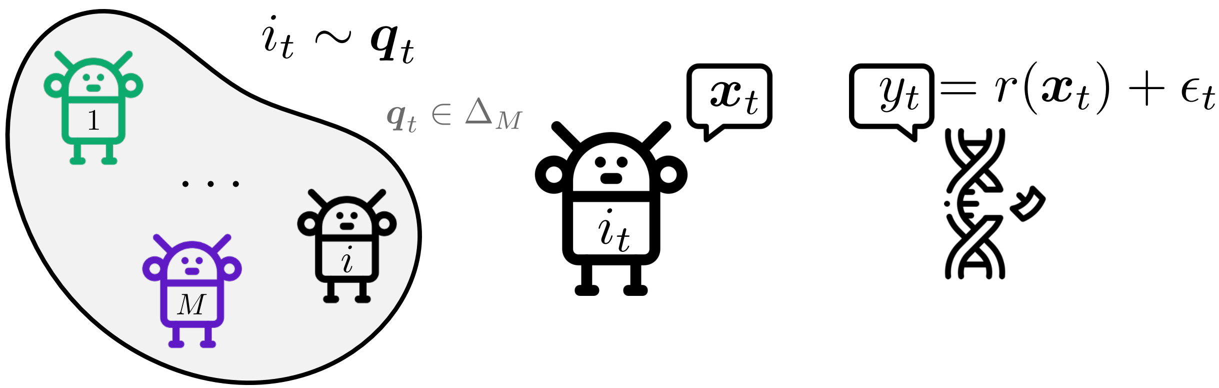


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Requires having observed the reward for the choice of each agent

Reward not observed? Hallucinate it.

image source: flaticon

# How to hallucinate rewards



[KEKP 2023]

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Turn group lasso into a sparse online regression oracle

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**Theorem (Anytime Lasso Conf Seq)**

*For appropriate choice of* $(\lambda_t)_{t\geq 1}$,

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Hallucinate the reward of agent j as

$$\hat{r}_{t,j} = \mathbb{E}_{\boldsymbol{x}\sim p_{t,j}} \hat{\boldsymbol{\theta}}_t^\top \boldsymbol{\phi}(\boldsymbol{x})$$

$p_{t,j} \in \mathcal{M}(\mathcal{X})$ action selection strategy

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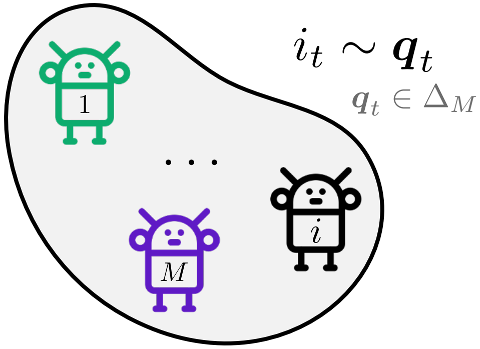


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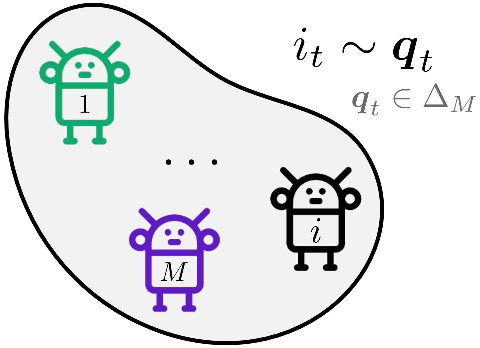


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## Exponential Weighting

$$q_{t,j} = \frac{\exp(\eta_t \sum_{s=1}^{t-1} \hat{r}_{s,j})}{\sum_{i=1}^{M} \exp(\eta_t \sum_{s=1}^{t-1} \hat{r}_{s,i})}$$

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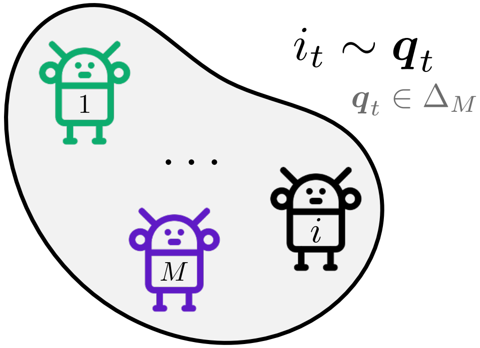


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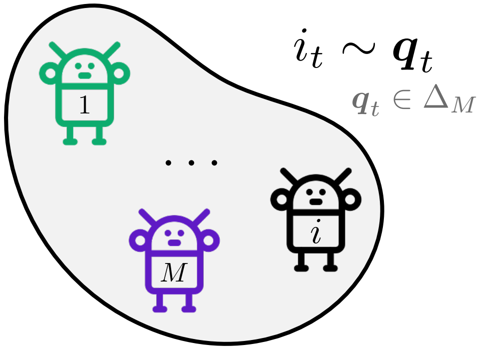


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sensitivity of updates

$$\hat{r}_{t,j} = \mathbb{E}_{\boldsymbol{x} \sim p_{t,j}} \hat{\boldsymbol{\theta}}_t^\top \boldsymbol{\phi}(\boldsymbol{x})$$

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**A**nytime **Exp**onential weighting algorithm with **L**asso reward estimates

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**Algorithm 1** ALEXP

Inputs: $\gamma_t$, $\eta_t$, $\lambda_t$ for $t \geq 1$

**for** $t \geq 1$ **do**

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**Theorem (Online Model Selection)**

*For appropriate choices of parameters, ALEXP with a UCB oracle agent satisfies*

$$R(T) = \mathcal{O}\left(\sqrt{T \log^3 M} + T^{3/4}\sqrt{\log M}\right)$$

*w.h.p. simultaneously for all $T \geq 1$.*

[Open problem of Agarwal et al. 2017 in the Linear case]

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Probably not tight? Lower bounds not clear.

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... is almost an Exp4 algorithm.

each expert is adaptive

as oppose to static experts with pre-set sequence of actions/advices

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as oppose to Importance Weighted Estimator or OLS

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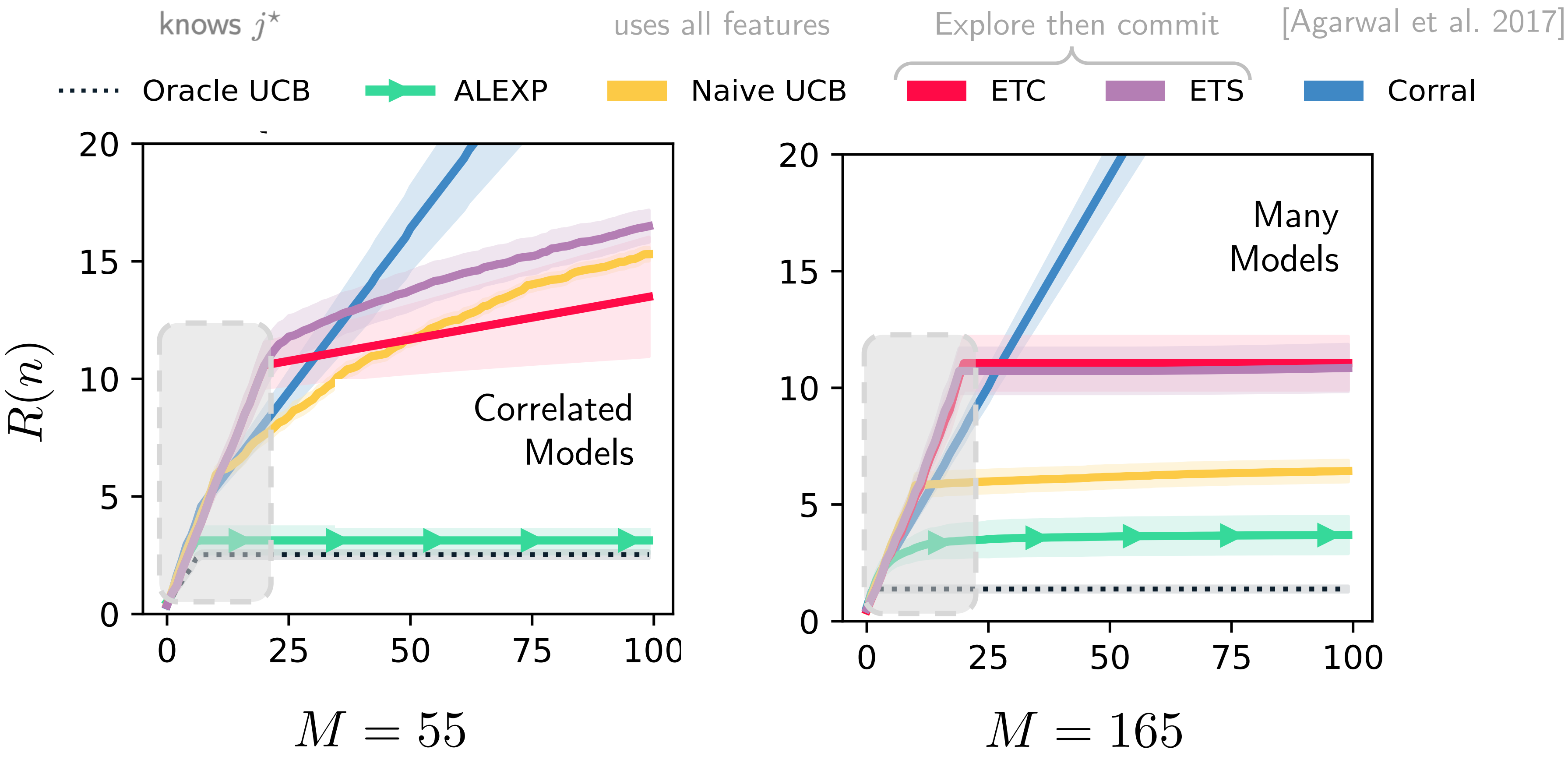
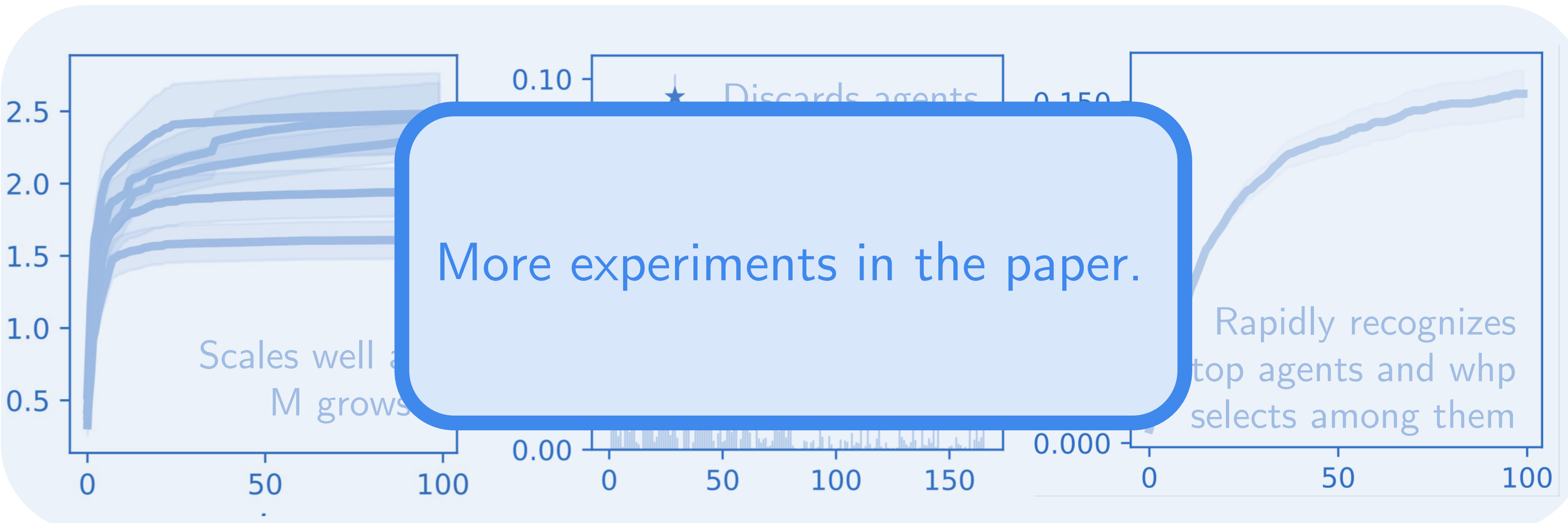
| $\hat{\boldsymbol{\theta}}_t \in \mathbb{R}^{dM}$ | $\mathrm{Var}^2(\hat{\boldsymbol{\theta}}_t)$ | $\mathrm{Bias}(\hat{\boldsymbol{\theta}}_t)$ |
| --- | --- | --- |
| OLS/IW | $M$ | $0$ |
| Lasso | $\log M$ | $\log M$ |

# Model Selection for Optimistic algorithms

data generation & baselines
described in the paper.



If I am running out of time:




More experiments in the paper.

If not...

# Model Selection Dynamics of ALExp

Let's see how things evolve turing training...

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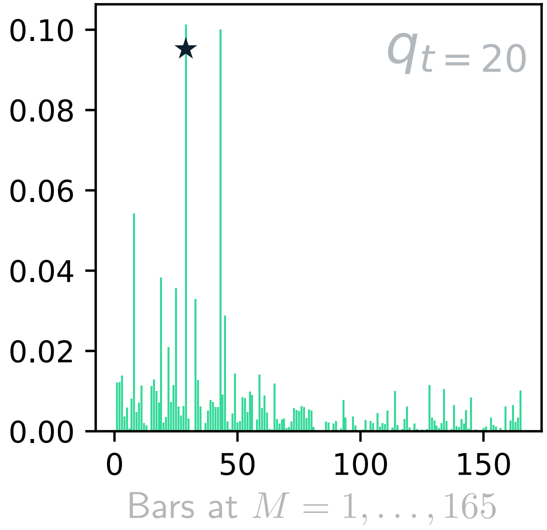
Distribution over the models
at time t=20



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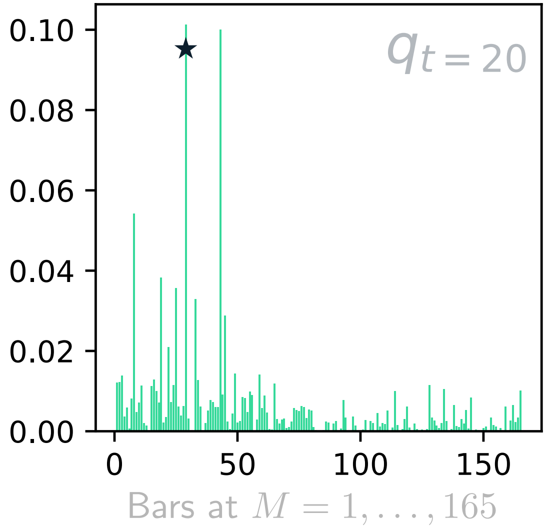


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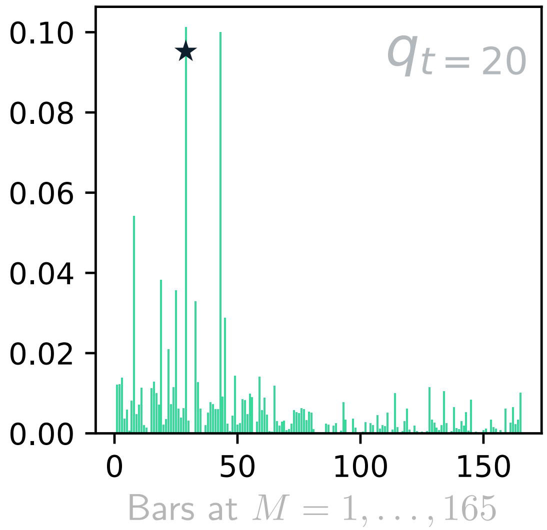
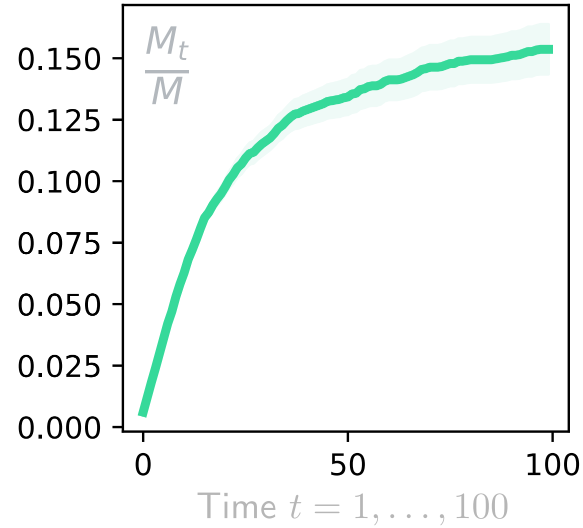



Discards agents without
having queried them

Number of visited agents
Total number of agents




Rapidly recognizes top agents
and whp selects among them

# What's left open?

1. Is exploration necessary for model selection?

$$\gamma_t \sim t^{-1/4}$$

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  Update selection distribution

connected to lowerbounds on min-eigenvals of covariance matrix

some new results: *pure* exploration is not necessary.

# What's left open?

1. Is exploration necessary for model selection?

2. For what other model classes (efficient) model selection is possible?

Linear ✔

$$\{\boldsymbol{\phi}_j : \mathbb{R}^{d_0} \to \mathbb{R}^d,\ j = 1, \dots, M\}$$

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Blackbox Class of size M?

$\mathrm{Poly}(M)$ lower bound?

Infinite class with bounded eluder dimension?

$\log \tilde{d}$ upper bound?

**PK**, Nicolas Emmenegger, AK, and Aldo Pacchiano. "Anytime Model Selection in Linear Bandits." NeurIPS, 2023.

# Thank you!

# How to hallucinate rewards

$$\hat{r}_{t,j} = \mathbb{E}_{\boldsymbol{x}\sim p_{t,j}} \hat{\boldsymbol{\theta}}_t^\top \boldsymbol{\phi}(\boldsymbol{x})$$

Turn lasso into a sparse online regression oracle

$$\hat{\boldsymbol{\theta}}_t = \arg\min \frac{1}{t} ||\boldsymbol{y}_t - \Phi_t \boldsymbol{\theta}||_2^2 + \lambda_t \sum_{j=1}^{M} ||\boldsymbol{\theta}_j||_2$$

**Theorem (Anytime Conf. Seq.)**

*If for all $t \geq 1$*

cost of going 'time uniform'

$$\lambda_t \geq \frac{c_1}{\sqrt{t}} \sqrt{\log(M/\delta) + \sqrt{d\,(\log(M/\delta) + (\log\log d)_+)}}$$

*then,*

$c_1$ and $c_2$ made exact in the paper

$$\mathbb{P}\left(\forall t \geq 1 : \left\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_t\right\|_2 \leq \frac{c_2 \lambda_t}{\kappa^2(\Phi_t, 2)}\right) \geq 1 - \delta$$

Restricted Eigenvalue property [check paper]

Variance & bias are both $\log M$

Difference with offline Lasso?

Instead of sub-gaussian concentration,

Design a self-normalized martingale based on $\left|\left|\left(\Phi_t^\top \boldsymbol{\epsilon}_t\right)_j\right|\right|$ (Empirical process error)

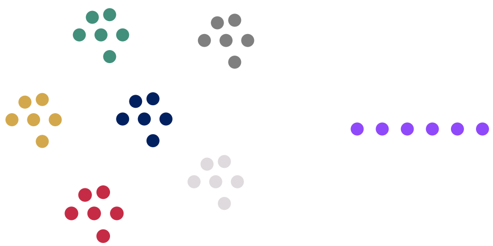
Apply a "stitched" time uniform boundary [Howard et al. '21]



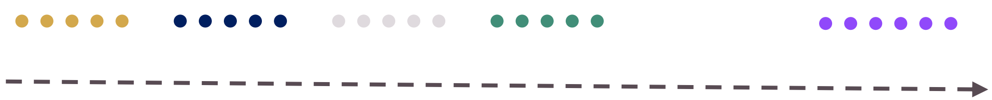
Thanks to Aaditya & SAVI friends!

# We consider 3 scenarios of increasing diffculty

1. Offline Data from similar tasks is available [KRK 2022]



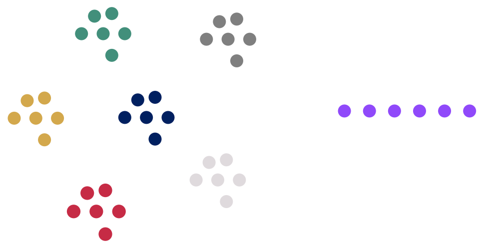
2. Online data from similar tasks can be available [SKRK 2023]



3. No data from similar tasks is available [KPEK 2023]

# Meta-Model Selection: Offline

When offline data from similar tasks is available,



$$y_{s,i} = r_s(\boldsymbol{x}_{s,i}) + \varepsilon_{s,i} \qquad i = 1, \dots, n \text{ and } s = 1, \dots, m$$

$$r_s(\cdot) = \sum_{j=1}^{M} \langle \boldsymbol{\theta}_s^{(j)}, \boldsymbol{\phi}_j(\cdot) \rangle \qquad J \text{ is shared}$$

Classical feature selection with Lasso

$$\hat{\boldsymbol{\theta}}^{(1)}, \dots, \hat{\boldsymbol{\theta}}^{(M)} = \arg\min \frac{1}{mn} \|\boldsymbol{y} - \sum_{j=1}^{M} \Phi_j \boldsymbol{\theta}^{(j)}\|_2^2 + \lambda \sum_{j=1}^{M} ||\boldsymbol{\theta}^{(j)}||_2$$

$$\hat{J} = \{j \in [M] \text{ s.t. } \hat{\boldsymbol{\theta}}^{(j)} > \omega\}$$

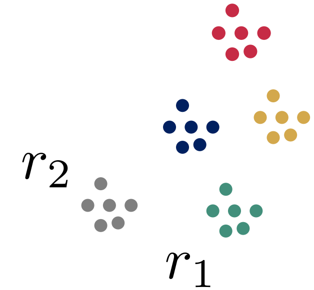
Solving the online optimization problem using the learned model

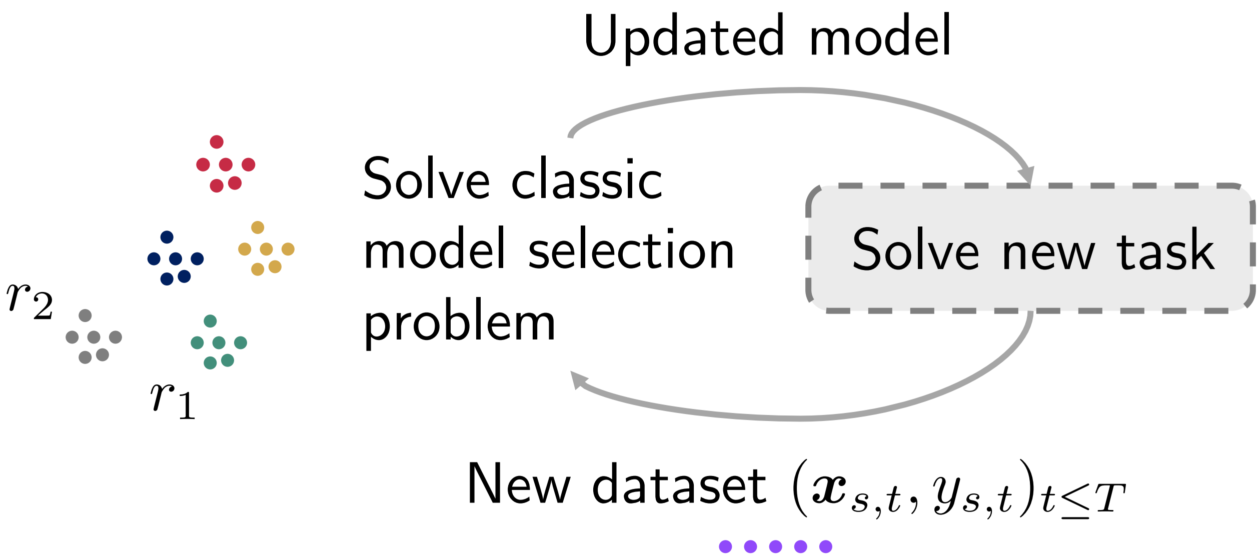
# Meta Model Selection: Lifelong

$\forall s \geq 1 : r_s \in \mathcal{H}$

Suppose the bandit task is of repetitive nature,

Optimizing for different molecular properties
Recommending products to different costumers



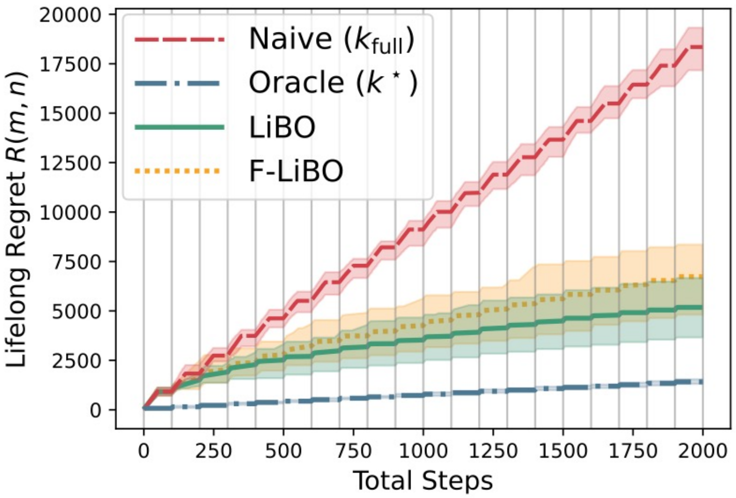



**Theorem (Lifelong Model Selection)**

*Under mild assumptions on the meta-data, and for an appropriate choice of $\lambda$, w.h.p.*

- *$\hat{J}$ is a consistent estimator of $J$,*
- *The optimization algorithm which uses $\hat{J}$ achieves oracle performance $R^\star(T, m)$,*

*as $m$ grows.*




the regret converges at a $\mathcal{O}(\log M/\sqrt{m})$ rate

$$R(T, m) = \sum_{s=1}^{m} \sum_{t=1}^{T} r_s(\boldsymbol{x}_s^\star) - r_s(\boldsymbol{x}_{s,t})$$